

Complex Geometry Exercises

Week 7

Exercise 1. Let (X, g, J) be a Kähler manifold and $\iota : Y \hookrightarrow X$ a closed complex submanifold. Use the Wirtinger inequality to prove that Y is volume-minimising within its homology class.

Exercise 2. Consider the open unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. Prove that the $(1, 1)$ -form

$$\omega = -\frac{i}{2} \partial \bar{\partial} \log(1 - |z|^2)$$

defines a Kähler metric on $\mathbb{D}$.

Exercise 3. The goal of the exercise is to prove that the Iwasawa manifold $\mathbb{I} = \mathbb{U}/\mathbb{U}_{\mathbb{Z}}$ is not Kähler.

- (i) Prove that holomorphic forms on a Kähler are harmonic.
- (ii) Compute a basis of left-invariant holomorphic 1-forms for $\mathbb{I}$ and conclude it is a weak Calabi-Yau manifold.
- (iii) Deduce it does not admit a compatible Kähler metric.

Exercise 4. Consider the hermitian metric

$$h_z(s, s') = \frac{\langle s, s' \rangle}{\sum_i |z_i|^2}$$

on $\mathcal{O}(1)$, induced by its global sections. Denote by ∇ its Chern connection. Prove

- (i) $\frac{i}{2\pi} F_{\nabla} = \omega_{FS}$, where ω_{FS} is the Kähler form of the Fubini-Study metric.
- (ii) $\int_{\mathbb{CP}^n} \omega_{FS}^n = 1$

(continues on the back)

Exercise 5. *Show that, in dimension ≥ 2 , there exists at most one Kähler metric in a given conformal class (up to scale).*

Exercise 6. *Let $\text{Pic}^0(X) := \ker(\text{Pic}(X) \rightarrow H^2(X, \mathbb{Z}))$. Prove*

(i) Show that if X is Kähler is a complex torus of dimension $b^1(X)$.

(ii) Give a counterexample when X is not Kähler.